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Two-Body Problem for Weber-Like Interactions

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The problem of two moving bodies interacting through a Weber-like force is presented. Trajectories are obtained analytically once relativistic and quantic considerations are neglected. The main results are that in the case of limited trajectories, in general, they are not closed and in the case of open trajectories, the deflection angles are not the same for similar particles with given energies and angular momenta but opposite potentials. This last feature suggests the possibility of a direct verification of the validity of Weber's law of force for electromagnetic interactions.

1. INTRODUCTION

The two-body problem is a classical one in physics; its resolution depends on the interacting force between the two bodies. A classical example is the case of central forces depending on the inverse square of the distance between the two bodies. Kepler's laws and Rutherford's differential scattering cross section are widely known results (Symon, 1978). However, it is also widely accepted that inverse square laws are strictly valid only when the bodies are not in relative motion with respect to each other. In electromagnetism, for example, if we want to treat the problem of two charged bodies in motion, retarded potentials should be used. This implies that in order to solve exactly the problem, all the previous history of motion should be known, and the conclusion is that the problem cannot be solved exactly; we can only approximate the solution in certain cases. This is due to the fact that we do not know what Gauss called "the keystone of electrodynamics" (Gauss, 1867), i.e., the true law of interaction between two moving charges.

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A possible law of interaction between two moving charges was proposed by Weber (1846). Weber's law of force has some interesting features: it reduces to Coulomb's law when the charges are at rest; it satisfies Newton's action and reaction principle; it can be derived from a velocity-dependent potential; it is completely relational, since it depends on the relative distance, velocity, and acceleration of the moving charges, so it has the same value for any observer; Faraday's law can be derived from it; and also Ampère's law for the force between two current elements (Ampère, 1825) can be derived from it.

In spite of the renewed interest in Weber's law (Assis, 1989*a*,*b*; Assis and Clemente, 1990; Wesley, 1987*a*) and experimental verifications of Ampère's law versus Grasmann's/Biot–Savart law for current elements (Graneau, 1982, 1983, 1985, 1989*a*,*b*; Graneau and Graneau, 1985; Moyssides and Pappas, 1986; Nasilowski, 1985; Pappas, 1983; Wesley, 1987*b*), the two-body problem for Weber-like interactions has not been considered in the literature.

Recently, we considered the unidimensional problem of two charges interacting through a Weber-like force (Assis and Clemente, 1990), finding implications on the limiting velocity of the charges. Here, we want to treat the bidimensional problem of two moving bodies interacting through a Weber-like potential.

It will be shown that the problem can be solved analytically once nonrelativistic or quantic considerations are included. The results show differences with the classical problem of two bodies interacting through a Coulomb-like potential. The main results are the possibility of perihelion precession for limited trajectories and the difference in deflection angles between scattering of particles with the same energies and impact parameters but opposite potential energies. This last result suggests the possibility of performing some experiment to directly check the validity of Weber's law.

It is worth noting that the necessity of performing classical scattering calculations based on force laws different from Coulomb's was already pointed out by Abdelkader (1968). O'Rahilly (1965) in his famous book already found corrections to Rutherford's formula by using Ritz's law of force (Ritz, 1911). Other force laws are available (Brown, 1955; Moon and Spencer, 1954; Warburton, 1946); we have considered Weber's, since it allows for quite simple calculations.

2. TWO-BODY PROBLEM

Let us consider, from a classical point of view, two point bodies of masses $m_{1,2}$, located at $\mathbf{r}_{1,2}(t)$, interacting through a Weber-like force (cgs

Gaussian units will be used throughout):

$$\mathbf{F}_{1,2} = -\mathbf{F}_{2,1} = -U_0 \frac{\hat{r}}{r^2}\left(1 + \frac{r\ddot{r}}{c^2} - \frac{\dot{r}^2}{2c^2}\right) \tag{1}$$

where U_0 is a constant ($U_0 = q_1 q_2$ if electromagnetic interaction is considered), c is the velocity of light, $r = |\mathbf{r}_1 - \mathbf{r}_2|$, $\hat{r} = (\mathbf{r}_1 - \mathbf{r}_2)/r$, and the overdot signifies d/dt. Here $\mathbf{F}_{i,j}$ represents the force that particle j exerts on particle i.

Without loss of generality the motion of the two bodies can be studied in the center-of-mass frame by introducing a fictitious particle of reduced mass $\mu = m_1 m_2/(m_1 + m_2)$. Since $\mathbf{F}_{1,2}$ is a central force, in the center-of-mass frame the angular momentum will be conserved. Introducing in the plane of motion a polar coordinate system r, θ, with origin at the center of mass, the conserved angular momentum can be expressed as

$$L = \mu r^2 \dot{\theta} \tag{2}$$

In this work we will restrict ourselves to $L \neq 0$, since $L = 0$ was already considered in another work (Assis and Clemente, 1990).

The energy of the reduced-mass particle in the center-of-mass frame will also be conserved during interaction. It can be shown that the Weber force can be deduced from a potential (Wesley, 1987*a*) and the following expression for the conserved energy arises:

$$W = \frac{\mu}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{U_0}{r}\left(1 - \frac{\dot{r}^2}{2c^2}\right) \tag{3}$$

where the first term on the rhs is the kinetic energy and the second is Weber's generalized potential energy.

Introducing $x^2 = 1 - K/r$ with $K = U_0/\mu c^2$ [here the restriction $W < \mu c^2(1 + c^2L^2/2U_0^2)$, arising from the vanishing of the potential energy when $\dot{r} = \sqrt{2}c$, is necessary in order to keep $x^2 > 0$], it results from (3) using (2) that

$$\frac{dx}{d\theta} = \pm\frac{1}{2x^2}[(x_1^2 - x^2)(x^2 - x_2^2)]^{1/2} \tag{4}$$

where

$$x_{1,2}^2 = 1 + \frac{\mu K U_0}{L^2}\left[1 \pm \left(1 + \frac{2WL^2}{\mu U_0^2}\right)^{1/2}\right] \tag{5}$$

$x_{1,2}^2$ represent possible turning points for x; if we assume that at least one of them exists, the condition $W \geq -\mu U_0^2/2L^2$ has to be fulfilled.

In order to integrate equation (4), it is worth noting to distinguish two situations as follows.

2.1. $U_0 < 0$. Attractive Force

In this case $x^2 \geq 1$. If $-\mu U_0^2/2L^2 \leq W < 0 \rightarrow x_{1,2}^2 > 1$, this means that the trajectory will be limited between the two radii represented by $x_{1,2}^2$, i.e., $x_2^2 \leq x^2 \leq x_1^2$. If $W = 0 \rightarrow x_1^2 > 1$ and $x_2^2 = 1$, the trajectory is open with $1 \leq x^2 \leq x_1^2$. If $W > 0 \rightarrow x_1^2 > 1$ and $x_2^2 < 1$, this means an open trajectory with $1 \leq x^2 \leq x_1^2$. In all cases x_1^2 represents the radius of closest approach and if we take $\theta = 0$ when $x^2 = x_1^2$ it is possible to find

$$\theta^A = \pm 2 \int_x^{x_1} \frac{dx\, x^2}{[(x_1^2 - x^2)(x^2 - x_2^2)]^{1/2}} = \pm 2|x_1| E(\phi, k) \tag{6}$$

where $E(\phi, k)$ is the incomplete elliptic integral of the second kind, argument

$$\phi = \arcsin\left(\frac{x_1^2 - x^2}{x_1^2 - x_2^2}\right)^{1/2}$$

and parameter

$$k = \left(\frac{x_1^2 - x_2^2}{x_1^2}\right)^{1/2}$$

$(0 \leq \phi \leq \pi/2$ and $0 \leq k^2 \leq 1)$.

2.2. $U_0 > 0$. Repulsive Force

In this case $x^2 \leq 1$. If $W < 0 \rightarrow x_{1,2}^2 > 1$, then no physical motion is possible, since $\dot{x}^2 < 0$. If $W = 0 \rightarrow x_1^2 > 1$ and $x_2^2 = 1$, the only possibility is that the bodies are at rest at an infinite distance. If $W > 0 \rightarrow x_1^2 > 1$ and $x_2^2 < 1$, this means an open trajectory with $x_2^2 \leq x^2 \leq 1$ once the restriction $W \leq \mu c^2(1 + c^2 L^2/2U_0^2)$ is imposed in order to avoid negative values of x_2^2. The x_2^2 will represent the point of closest approach, and if we take $\theta = 0$ when $x^2 = x_2^2$, it is possible to obtain

$$\theta^R = \pm 2 \int_{x_2}^{x} \frac{dx\, x^2}{[(x_1^2 - x^2)(x^2 - x_2^2)]^{1/2}} = \pm 2|x_1|[E(k) - E(\phi, k)] \tag{7}$$

where $E(\phi, k)$ represents the incomplete elliptic integral of the second kind, ϕ and k being the same as above, and $E(k)$ is the complete elliptic integral of the second kind.

Expressions (6) and (7) formally solve the problem of the trajectory of two bodies interacting through a Weber-like force. It is worth noting that classical results due to a Coulomb-like force can be recovered by properly taking the limit $c \rightarrow \infty$ in formulas (6) and (7). In this case $|x_1| \rightarrow 1$ and $k \rightarrow 0$ in such a way that $E(\phi, k) \rightarrow \phi$ and $E(k) \rightarrow \pi/2$.

3. DISCUSSION AND CONCLUSIONS

It is convenient to divide the discussion into two parts, limited and open trajectories

3.1. Limited Trajectory

This occurs when the force is attractive and $W<0$. Excluding the special case in which $x_1^2=x_2^2$, which represents a circular orbit perfectly equivalent to the case of simple Coulomb-like interaction, in general, the orbit will be comprised between two turning radii defined by $x_{1,2}^2$. Such radii are the same, for given energy and angular momentum, as in the case of Coulomb-like interaction. What is different is that the trajectory is not a closed ellipse. In this respect it is interesting to calculate the angle described by the trajectory when the reduced-mass particle goes from the perihelion, reaches the aphelion, and returns to the perihelion. Such an angle is

$$\Delta\theta = 4|x_1|E(k) \tag{8}$$

It is always greater than 2π. Assuming $r_{1,2}=a(1\mp\varepsilon)$ the perihelion and aphelion radii (a and ε can be interpreted as the semimajor axis and eccentricity of the ellipse approximating the orbit), the shift in the perihelion of the orbit after one cycle can be easily calculated in the limit of small $|K|$:

$$\delta\theta \simeq \frac{\pi|K|}{a(1-\varepsilon^2)} \tag{9}$$

This result was already obtained by Assis (1989*b*), where a Weber-like law for gravitational interaction was proposed in order to explain inertia, by solving the linearized equations of motion instead of linearizing the exact solution. The correspondence with the motion of the perihelion of Mercury, in accordance with general relativity, is obtained when $U_0 = -Gm_1m_2$ (G being the universal gravitational constant) and c^2 in equation (1) is replaced by $c^2/6$. It is worth noting that $\Delta\theta - 2\pi$, with $\Delta\theta$ given by (8), represents the perihelion shift to all orders in $|K|$.

3.2. Open Trajectory

This occurs when $W \geq 0$, independent of the sign of U_0. In analogy with the classical Rutherford scattering problem, where the angle of deflection α of a reduced-mass particle with a given energy $W=\mu v_0{}^2/2$ and impact parameter s, such that $L=\mu v_0 s$, is given by

$$\sigma = 2\arctan(S^2)^{1/2} \tag{10}$$

where $S^2=4W^2s^2/U_0{}^2$, we calculate the corresponding deflection angle α^A for the attractive case as

$$\alpha^A=4|x_1|E(\phi^*,k)-\pi=\frac{4E(\phi^*,k)}{(1-k^2\sin^2\phi^*)^{1/2}}-\pi \tag{11}$$

where $\sin^2\phi^*=(x_1^2-1)/(x_1^2-x_2^2)$ and $k^2=(x_1^2-x_2^2)/x_1^2$. Analogously, for the repulsive case,

$$\alpha^R=\pi-4\frac{E(k)-E(\phi^*,k)}{(1-k^2\sin^2\phi^*)^{1/2}} \tag{12}$$

where ϕ^* and k are the same as for α^A. It can be seen that α^A and α^R are functions of S^2 (through ϕ^* and k^2) and $v_0{}^2/c^2$ (through k^2), and they do not coincide. This is perhaps the most interesting feature; while in Rutherford scattering there is no difference in α when the sign of U_0 is reversed, for Weber-like interactions two different deflection angles result. Moreover, while $\alpha^R\leq\pi$, as is the case for α, α^A has no limit; it diverges when $S^2\to 0$, since $\phi^*\to\pi/2$ and $k^2\to 1$. The difference $\alpha^A-\alpha^R$ is an increasing function of $v_0{}^2/c^2$. As an example, we show in Figure 1 α^A and α^R as functions of S^2 for the case of $v_0{}^2/c^2=0.2$; α has not been shown, since it is too close to α^R in order to appreciate the difference.

It is also interesting to compare the resulting scattering differential cross sections

$$\frac{d\sigma}{d\Omega}=\frac{2\pi s\,ds}{2\pi\sin\alpha\,d\alpha}$$

In terms of S^2 we have in the Rutherford case

$$\frac{d\sigma}{d\Omega}=\frac{U_0}{16W^2}\left|\frac{2\,dS^2}{d(\cos\alpha)}\right|=\frac{U_0}{16W^2}\left(\sin^4\frac{\alpha}{2}\right)^{-1} \tag{13}$$

For the Weber attracting case:

$$\left(\frac{d\sigma}{d\Omega}\right)^A=\frac{U_0}{16W^2}\left|\frac{2\,dS^2}{d(\cos\alpha^A)}\right| \tag{14}$$

and for the Weber repulsive case:

$$\left(\frac{d\sigma}{d\Omega}\right)^R=\frac{U_0}{16W^2}\left|\frac{2\,dS^2}{d(\cos\alpha^R)}\right| \tag{15}$$

In Figure 2, expressions (13)-(15) have been plotted in $U_0{}^2/16W^2$ units as functions of the deflection angle and $v_0^2/c^2=0.2$. As can be seen, a small difference exists between expressions (13) and (15), but expression (14) strongly differs from the other at α close to π. In the Weber attracting case

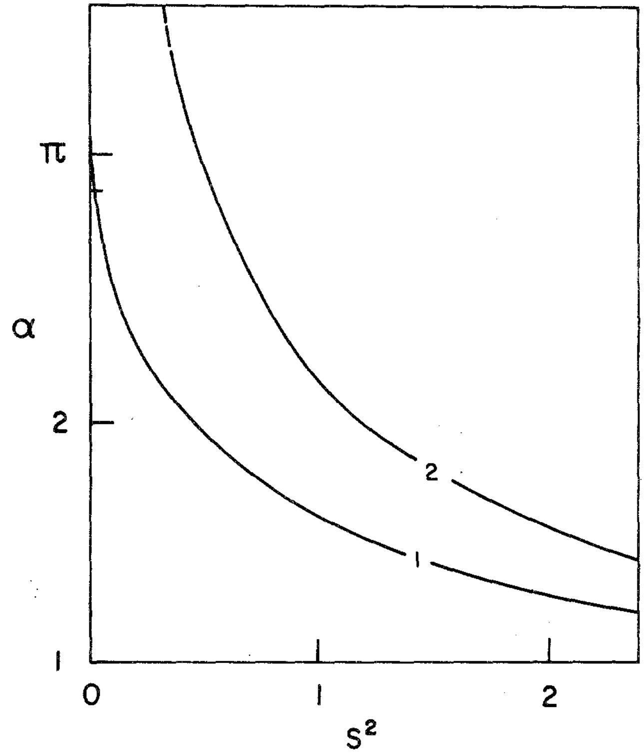

Fig. 1. Deflection angles as a function of $S^2 = 4W^2s^2/U_0^2$ for $v_0^2/c^2 = 0.2$ and Weber repulsive (curve 1) and attractive (curve 2) interactions.

the pole in the differential scattering cross section always exists at $\alpha = \pi$; the departure from the other curves is an increasing function of v_0^2/c^2.

In conclusion, the problem of two bodies interacting through a Weber-like force has been solved analytically from a classical point of view. As for Coulomb-like interactions, limited and open trajectories have been found depending on the energy of the system. Limited trajectories occur in the case of attractive force and negative energy; they differ from common ellipses since in general they are not closed curves. In this respect an expression for the precession of the perihelion has been obtained.

For open trajectories, perhaps the most interesting feature is that, once the energy and the impact parameter are assigned, the deflection angle is not the same for attractive and repulsive forces, as was the case in Rutherford scattering. In attractive scattering the deflection angle is not limited when $s \to 0$, while in the repulsive case it tends to π as in the Rutherford case. This implies strong differences in the differential scattering cross sections,

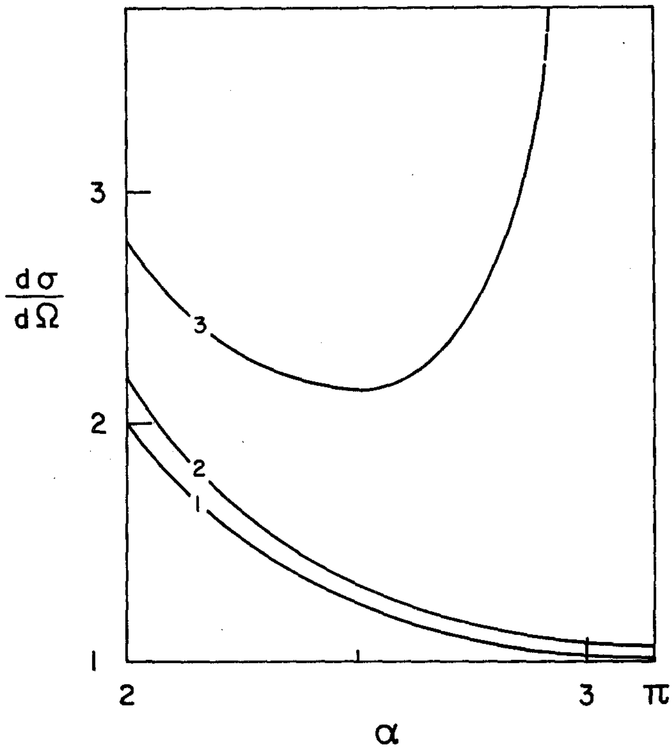

Fig. 2. Differential scattering cross sections in $U_0^2/16\,W^2$ units as a function of the deflection angle and $v_0^2/c^2 = 0.2$. Curve 1 represents the Rutherford case, curves 2 and 3 the Weber repulsive and attractive case, respectively.

which increase when the energy increases. This aspect suggests the possibility of an experimental check of the validity of Weber's law for electromagnetic interactions; perhaps it would be possible to detect the difference in scattering angle of electron and positron beams interacting with some blanket.

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